Time evolution of continuous-time quantum walks on dynamical percolation graphs

Zoltán Darázs*
Eötvös University, Pázmány Péter sétány 1/A, H-1117 Budapest, Hungary and
WIGNER RCP, SZFI, Konkoly-Thege Miklós út 29-33, H-1121 Budapest, Hungary

Tamás Kiss
WIGNER RCP, SZFI, Konkoly-Thege Miklós út 29-33, H-1121 Budapest, Hungary

We study the time evolution of continuous-time quantum walks on randomly changing graphs. At certain moments edges of the graph appear or disappear with a given probability. We focus on the case when the time interval between subsequent changes of the graph tends to zero. We derive explicit formulae for the general evolution in this limit. We find that the percolation in this limit causes an effective time rescaling. Independently of the graph and the initial state of the walk, the time is rescaled by the probability of keeping and edge. Both the individual trajectories for a single system and average properties with a superoperator formalism are discussed. We give an analytical proof for our theorem and we also present results from numerical simulations of the phenomena for different graphs.

PACS numbers: 03.67.Ac, 05.40.Fb

I. INTRODUCTION

Randomness in graph theory has been an ever developing field since the pioneering works by Erdős and Rényi [1]. Today, there is a refined mathematical theory for treating random graphs [2] and, at the same time, statistical methods borrowed from physics continue to be a motor for research in random network theory [3]. Applications of classical random graph theory often involve the notion of dynamics on the random graph, describing phenomena ranging from disease spreading [4] to percolation [5]. Random walks provide a simple, fundamental model for dynamics on graphs and random graphs, being one of the first problems in random graph theory and later used throughout its applications [6].

Quantum dynamics, and especially quantum walks on graphs have been a topic of much interest in the past few years [7, 8]. The concept of quantum walk was first introduced by Aharonov *et al.* [9] as a discrete-time dynamical system. The continuous-time quantum walk was shown to be an effective tool for quantum computation in the pioneering paper by Farhi and Gutmann [10]. Possible applications in quantum information processing triggered thorough research on all aspects of quantum walks.

Apart from their role as a universal quantum information processing primitive [11], continuous-time quantum walks are a conceptually simple, but effective model for transport [12]. Randomization of the underlying graph structure is motivated by uncontrolled processes in the physical model, i.e. transport in disordered media [13], similar to percolation in a classical system. The percolation quantum walk problem has also been considered for discrete time quantum walks [14].

In the theory of static percolation, we have a given graph, in which we can keep an edge with probability λ, or remove it with probability $1-\lambda$. During the time evolution, we keep the graph constant. The ensemble average of a quantity of the system can be calculated as an average over the possible realizations of the percolation graph. In case of continuous-time quantum walks, the static percolation problem has already been considered under the name: statistical networks [15, 16].

For quantum walks, changing the underlying graph during the time evolution randomly at some rate leads to a different problem. In the context of percolation, this is called the dynamical percolation problem [17]. Dynamical percolation for discrete time quantum walks has been studied numerically [14] and its asymptotic behavior for one dimensional systems also analytically [18].

In this paper we consider the problem of continuous-time quantum walk on a dynamical percolation graph. First, we define the problem of dynamical percolation for continuous-time quantum walks. Unlike in the discrete time case, there is no natural timing for changing the graph structure. One has to introduce a time step for changes. Then we concentrate on the continuous percolation limit, when changes in the graph may happen infinitely frequently.

The structure of the paper is the following. In Section II. we briefly review the basic properties of the continuous-time quantum walk. Then in Section III. we consider the time evolution of the walk on a percolation graph. We introduce a simple formula for the time evolution and we analitycally prove it. We extend our result for ensemble average of quantum walks on percolation graphs with a superoperator formalism in Section IV. Finally, in Section V. we show some numerical examples to demonstrate behavior for finite parameters.

*Electronic address: darazs.zoltan@wigner.mta.hu

II. CONTINUOUS-TIME QUANTUM WALK

Quantum walks have two main types: the discrete- and the continuous-time case. In the discrete-time quantum walk the walker moves on a discrete graph, and changes its position in discrete time intervals [19, 20].

The concept of continuous-time quantum walk was defined by Farhi and Gutmann [10]. We have a $G(V, E)$ undirected graph, and let us denote the vertices with $a = 1 \dots N$. Then we construct an N-dimensional Hilbert space with an orthogonal $\{|a\rangle\}$ basis corresponding to the graph where $a = 1 \dots N$ and $\langle a | b \rangle = \delta_{a,b}$. The unitary time evolution reads

$$\hat{U}(t) = e^{-i\hat{H}t}\,, \tag{1}$$

where the matrix elements of the Hamiltonian are

$$\langle a|\hat{H}|b\rangle = \begin{cases} -\gamma, & \text{if } |a\rangle \text{ and } |b\rangle \text{ are neighbouring states,} \\ 0, & \text{if } |a\rangle \text{ and } |b\rangle \text{ not neighbouring,} \\ k\gamma, & \text{if } |a\rangle = |b\rangle. \end{cases} \tag{2}$$

In the definition of $\hat{H}$ the k parameter is the degree of the vertex, and γ is a time-independent constant. For simplicity, we can choose $\gamma = 1$, it only rescales time. The wave function of the system after time t reads

$$|\Psi(t)\rangle = e^{-i\hat{H}t}|\Psi(0)\rangle\,. \tag{3}$$

The probability, that we measure the walker at the state $|b\rangle$ after time t if we started the walk from the state $|a\rangle$ can be calculated as

$$\pi_{ba}(t) = \left|\langle b|\, e^{-i\hat{H}t}\, |a\rangle\right|\,. \tag{4}$$

III. QUANTUM WALK ON A PERCOLATION GRAPH

In the previous sections we briefly introduced the basic model of percolation, and the continuous-time quantum walk. In this section we consider how the dynamics of the continuous-time quantum walk changes if we modify the underlying graph randomly during the walk. We focus on the case when the time interval between the changes tends to zero.

Let us suppose that we have a graph with N edges, and the nodes are connected by a given starting geometry (e.g. ring). Then in given time steps we change the underlying graph randomly. In each step we can keep an edge of the original graph with probability λ, or discard it with $1 - \lambda$. We evolve the system for time T with step size τ. In this case the number of the steps is $S = T/\tau$.

For every possible realization of the graph, we can introduce the

$$\hat{U}_r(\tau) = e^{-i\hat{H}_r\tau} \tag{5}$$

unitary operator according to (1), where the number of the r realization indexes is $R = 2^N$.

Let us suppose that the initial state of the system is $|\psi(0)\rangle$. In each step, we act on the actual state of the system with an $\hat{U}_r(\tau)$ unitary operator. Introduce the following notation: let $\hat{U}_{r_s}(\tau)$ be the time evolution operator according to the randomly generated graph in the sth step, in accordance with (2). With this notation, the state at time T, after S steps reads

$$\begin{aligned} |\psi(T)\rangle &= \hat{U}_{r_S}(\tau) \dots \hat{U}_{r_2}(\tau)\hat{U}_{r_1}(\tau)\,|\psi(0)\rangle \\ &= e^{-i\hat{H}_{r_S}\tau} \dots e^{-i\hat{H}_{r_2}\tau} e^{-i\hat{H}_{r_1}\tau}\,|\psi(0)\rangle\,. \end{aligned} \tag{6}$$

Since the H_r operators do not commute, we can use the Zassenhaus product formula:

$$\begin{aligned} e^{t(A+B)} &= e^{tA}e^{tB}\prod_{j=2}^{q} e^{t^j C_j^{(A,B)}} + \mathcal{O}(t^{q+1})\,, \\ \lim_{n\to\infty} e^{tA_1}e^{tA_2}\dots e^{tA_p}e^{t^2 C_2^{\{A_j\}}} &\dots e^{t^n C_n^{\{A_j\}}} = e^{t\sum_{j=1}^{p} A_j}\,, \end{aligned} \tag{7}$$

where the $C_j^{(A,B)}$, $C_n^{\{A_j\}}$ parameters are the Zassenhaus exponents [21, 22]. We are interested in the $\tau \to 0$ limes, therefore we will neglect the higher order terms in τ:

$$e^{-i\hat{H}_{r_p}\tau}e^{-i\hat{H}_{r_q}\tau} = e^{-i\tau(\hat{H}_{r_p}+\hat{H}_{r_q})} + \mathcal{O}(\tau^2)\,. \tag{8}$$

With this approximation, if τ is small enough, the final state reads

$$|\psi(T)\rangle = \left(e^{-i\tau\sum_{s=1}^{S}\hat{H}_{r_s}} + \mathcal{O}(\tau^2)\right)|\psi(0)\rangle\,. \tag{9}$$

Let us consider the probabilities of the realizations. The probability of the realization that contains N edges is $p^{(N)} = \lambda^N$. The probability of a realization, in which one edge is missing is $p^{(N-1)} = \lambda^{N-1}(1-\lambda)$, and this case is degenerate, because the number of the graphs that contain $N-1$ edges is N. It can be seen that the probability of a configuration that contains k edges is $p^{(k)} = \lambda^k(1-\lambda)^{N-k}$, and the degeneracy is $\binom{N}{k}$.

We can decompose every H_r matrix in the following way

$$H_r = \sum_{k\in\mathcal{E}_r} E_k\,, \tag{10}$$

where $\mathcal{E}_r$ is the set of the k values according to the edges present in the rth realization, and E_k is a matrix belonging to one edge in the graph. For example, if the H_r realization contains two edges, the first and the second one, then $H_r = E_1 + E_2$, and the Hamiltonian of the graph without percolation reads

$$H = \sum_{k=1}^{N} E_k\,. \tag{11}$$

Our task is now to evaluate the sum

$$\mathbb{S} = \sum_{s=1}^{S} H_{r_s} \,. \tag{12}$$

If $S > R$ then some H_r matrix occurs more than once in $\mathbb{S}$. The probability that we get the H_r matrix x times from the S steps follows a binomial distribution. If $S \gg R$ then we can use the law of large numbers. We can estimate the number of occurrences of one H_r matrix in the $\mathbb{S}$ sum as $p_r S + \mathcal{O}(\sqrt{S})$, where p_r is the probability of realizing the graph that belongs to H_r. The accuracy of our estimation can be calculated from the standard deviation $\sqrt{S p_r (1-p_r)}$, it leads to an $\mathcal{O}(\sqrt{S})$ error.

If the realization contains k edges, then $p_r = p^{(k)}$. Now let us calculate, how many times a chosen E_k matrix occurs in the $\mathbb{S}$ sum. This matrix belongs to the k edge in the graph. Choose j edges, where $j = 1 \dots N-1$, and put them to the graph next to the k edge. We can do it in $\binom{N-1}{j}$ ways. Therefore the number of the H_r realizations, that contains E_k reads

$$\sum_{j=0}^{N-1} \binom{N-1}{j} . \tag{13}$$

A given H_r matrix occurs $p_r S + \mathcal{O}(\sqrt{S})$ times in the $\mathbb{S}$ sum, it means, that the sum contains the E_k matrix

$$\sum_{j=0}^{N-1} \binom{N-1}{j} \left[p^{(j+1)} S + \mathcal{O}(\sqrt{S}) \right] = \\ = S\lambda + \mathcal{O}(\sqrt{S}) \tag{14}$$

times. It is true for every k, therefore

$$\mathbb{S} = \sum_{s=1}^{S} H_{r_s} = \left[S\lambda + \mathcal{O}(\sqrt{S}) \right] H \,. \tag{15}$$

With this result the exponential in (9) can be written as

$$-i\tau \sum_{s=1}^{S} \hat{H}_{r_s} = -iT\hat{H} \left(\lambda + \mathcal{O}(\sqrt{\tau}) \right) . \tag{16}$$

Taking the $\tau \to 0$ limes, we arrive at our main result

$$|\psi(T)\rangle = \lim_{\tau \to 0} \prod_{k=1}^{\frac{T}{\tau}} \hat{U}_{r_k}(\tau) \, |\psi(0)\rangle = \hat{U}(T\lambda) \, |\psi(0)\rangle \,. \tag{17}$$

The effect of a finite time step τ will be examined numerically in a later section.

Finally one can ask what happens if we take the extremal values for the λ probability. Both cases are trivial. If $\lambda = 0$ then we have a null graph, there is no time evolution, the system stays in the initial state. If $\lambda = 1$ then we get the percolation free graph in each step, we get back the percolation free evolution. We can simply arrive at these results if we write $\lambda = 0$ or 1 into the (17) equation.

IV. SUPEROPERATOR FORMALISM

In this section we would like to consider the average properties of the randomly evolved systems that we introduced in the previous section. The standard way to calculate the ensemble average of a quantity of a system reads

$$\langle ... \rangle = \frac{1}{\mathfrak{T}} \sum_{\mathfrak{t}=1}^{\mathfrak{T}} [...]_{\mathfrak{t}} = \sum_{\mathfrak{t}=1}^{\mathfrak{T}} p_{\mathfrak{t}} [...]_{\mathfrak{t}} \,, \tag{18}$$

where $\mathfrak{T}$ is the number of the possible trajectories, and $\mathfrak{t}$ denotes one trajectory [12].

In each step, we can keep an edge with probability λ, or discard it. Let us denote the length of a time interval with τ. Let us consider a periodical chain with N nodes, and evolve the system until time T. In this case, the graph changes T/τ times. If $N > 2$, then the graph has N edges, therefore the number of possible graph realizations is $R = 2^N$. It means, that for time T the number of the possible trajectories is $\mathfrak{T} = (2^N)^{(T/\tau)}$, because we can choose from R graphs in each step. For a big graph with good time resolution this explicit evaluation becomes very difficult. For example the number of the trajectories for a periodical chain with $N = 15$ nodes with $S = 5000$ steps is over 10^{22577} (we use these parameters later for a numerical simulation with the superoperator formalism).

Instead of evaluating all possible trajectories one by one, we can use a superoperator formalism [23–25], in which we use the formerly introduced $\hat{U}_r(\tau)$ unitary operators. In one step, we act on the density operator with the

$$\hat{\phi}_\tau(\hat{\rho}) = \sum_{r=1}^{R} p_r \hat{U}_r(\tau) \hat{\rho} \hat{U}_r^\dagger(\tau) \tag{19}$$

superoperator, where p_r is the probability of the graph realization denoted by r as before, and we sum over all possible realizations. If we evaluate the system for time T with τ steps, then we should act $S = T/\tau$ times with the $\hat{\phi}_\tau$ superoperator, and the final state reads

$$\hat{\rho}(T) = \overbrace{\hat{\phi}_\tau(\hat{\phi}_\tau(\hat{\phi}_\tau(\dots \hat{\phi}_\tau}^{S}(\hat{\rho}))) = \phi_\tau^{(S)}(\hat{\rho}(0)) \,. \tag{20}$$

If we use the (5) form of $\hat{U}_r(\tau)$, the final state can be written as

$$\hat{\rho}(T) = \sum_{r_S=1}^{R} p_{r_S} \hat{U}_{r_S}(\tau) \Bigg(\sum_{r_{S-1}=1}^{R} p_{r_{S-1}} \hat{U}_{r_{S-1}}(\tau) \Bigg(... \\ ... \sum_{r_1=1}^{R} p_{r_1} \hat{U}_{r_1}(\tau) \hat{\rho}(0) \hat{U}_{r_1}^\dagger(\tau) ... \Bigg) \hat{U}_{r_{S-1}}^\dagger(\tau) \Bigg) \hat{U}_{r_S}^\dagger(\tau) = \\ = \sum_{r_1 \dots r_S=1}^{R} p_{r_1} ... p_{r_S} e^{-i\hat{H}_{r_S}\tau} ... e^{-i\hat{H}_{r_1}\tau} \hat{\rho}(0) e^{i\hat{H}_{r_1}\tau} ... e^{i\hat{H}_{r_S}\tau} \,. \tag{21}$$

Here we can use the same treatment in the $\tau \to 0$ limes that we use in the previous section. In that section we demonstrated, that in the $\tau \to 0$ limes the typical behaviour of the possible trajectories is only a time rescaling, therefore if we take the average of the trajectories we again get the time rescaled dynamics. Therefore the act of the superoperator can be written in a very simple form:

$$\hat{\rho}(T) = \lim_{\tau \to 0} \hat{\phi}_{\tau}^{\left(\frac{T}{\tau}\right)}(\hat{\rho}(0)) = \hat{U}(T\lambda)\hat{\rho}(0)\hat{U}^{\dagger}(T\lambda)\,. \qquad (22)$$

V. NUMERICAL SIMULATIONS

In the previous sections we demonstrated that, if we have a percolation graph in which we keep an edge with probability λ in every τ time step, then in the dynamics of a continuous-time quantum walk on this graph, the percolation only causes a time rescaling if the τ time interval tends to zero. In our calculation we use the $\tau \to 0$ limes. Of course, it is only a mathematical limes, in a physical system we cannot get his exact limes. In this section we show some concrete example to show, that for every time T there exists a small enough τ parameter, for which we get a very good estimation for the final state. This can be relevant for experimental realization of the phenomena.

In our simulations we calculate the probability that the walker returns to the origin. It is a perfect parameter to illustrate our analytical result, it characterizes the transport properties of the walk. Many articles use the return probability to investigate the properties of the continuous-time quantum walk [26–29].

In our first example we consider the continuous-time quantum walk on a percolated 2D integer lattice that contains $N = 100$ nodes. We address the nodes by i, j, it means the ith node in the jth column. We evolve the system with $\tau = 10^{-4}$ time steps in $S = 10^5$ steps. We start the system from the $|\Psi(0)\rangle = |45, 45\rangle$ state, and we calculate the probability of being at this node. We use different λ values, and we plot both the numerical and the theoretical values according to (17). The results are in Fig. 1. Incidentally, we note that the shape of the curves are very close to the appropriate Bessel function [30] if we scale them by the $T \to \lambda T$ transformation in the percolation case. These results suggest that our results can be extended for infinite 2D integer lattice graphs.

In the next example we demonstrate that the equation (22) gives a very good approximation for the time evolution with finite τ values if we take an average over the trajectories, i.e. we use the superoperator formalism. We take a periodical chain with $N = 15$ nodes, and we evolve the system with $\tau = 0.004$ steps from the $\hat{\rho}(0) = |1\rangle\langle 1|$ initial state in $S = 5000$ steps. We use different values for the probability, and we also plot the theoretical line from (22). Our result are in Fig. 2. Again, we can be see that the starting shape of the curves are very close to the $J_0^2(2t)$ Bessel function [31]. This fact again allows us to suggest that our results are valid for the infinite 1D integer lattice graph.

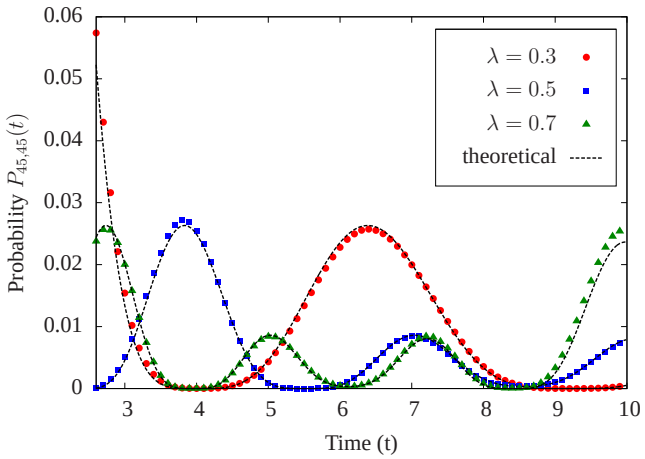

FIG. 1: The numerical simulation of one trajectories in a 10×10 2D lattice in 10^5 steps with different probabilities and the theoretical line.

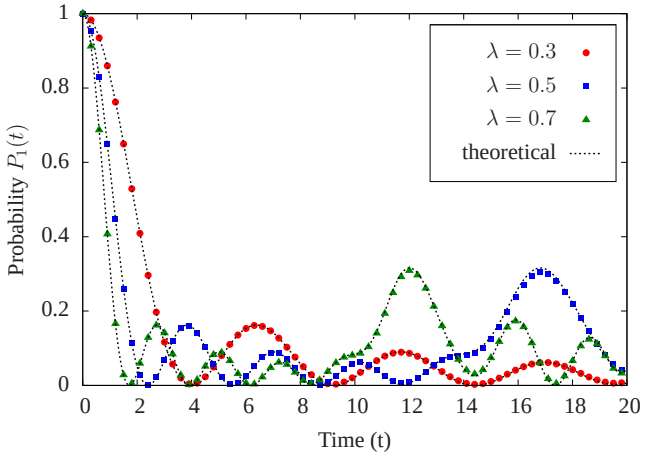

FIG. 2: The numerical simulation of the superoperator formalism in a periodical chain with $N = 15$ nodes. The parameters are $S = 5000$ and $\tau = 0.004$.

In the third figure we illustrate that the precision of the (22) formula can be arbitrary fine if we choose small enough τ. We consider the system for time $T = 10$, and we divide the $[0, T]$ interval into S part, i.e. we evolve the system in S step until T. Therefore if S is big, then the $\tau = T/S$ parameter is small. In our simulation we use a periodical chain with $N = 10$ nodes with different initial state and λ probabilities. For every S values, we calculate the probability of being at the $|1\rangle$ state both with the numerical simulation and the theory, and we take the maximum difference between them. Our results can be seen in Fig. 3. It is nicely seen that the difference between the simulation and the theory is a decreasing function of the number of the steps.

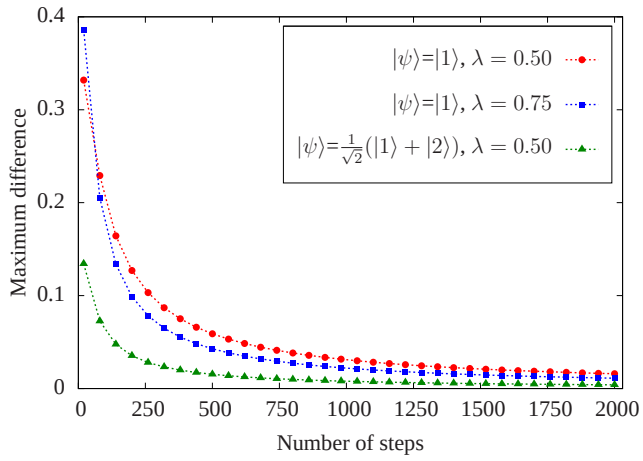

FIG. 3: The maximum difference between the numerical simulation and the (22) theoretical values of the probability of being at the $|1\rangle$ state in an $N = 10$ periodical chain. The time interval that was divided into S parts was $[0, 10]$.

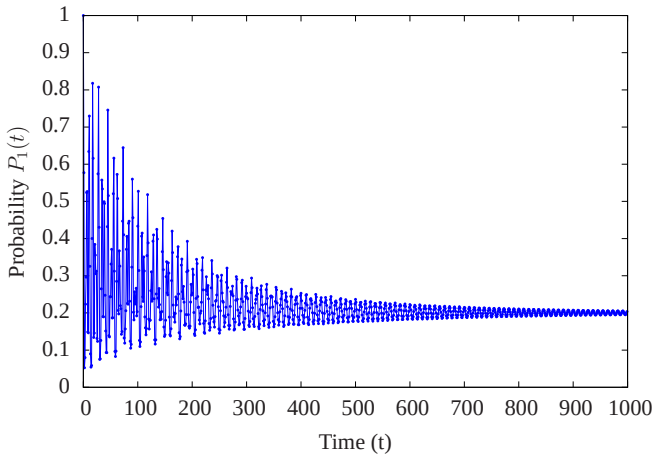

FIG. 4: The probability of measure the walker at the first node in an $N = 5$ periodical chain. We evolve the system for $S = 10^5$ steps with $\tau = 0.01$ time intervals. We choose the probability of keeping an edge for $\lambda = 0.5$.

In our last simulation we look at what happens, if we evolve the system for a long time with not too small τ time steps. We consider a periodical chain with $N = 5$ nodes for $S = 10^5$ steps with $\tau = 0.01$ time interval, $\lambda = 0.5$ probability, and we use the superoperator formalism. The initial state of the system was $\hat{\rho}(0) = |1\rangle\langle 1|$. With these settings we can see that after a long time (after lot of steps), the probability that we measure the walker at the initial state can be described with

$$P_1(t) = \frac{1}{N}\,. \tag{23}$$

This asymptotic dynamics has already been considered for the discrete-time quantum walk [18], and it seems that similar methods can be applied to the continuous-time case [32]. In our simulation we calculate the probability of being at the $|1\rangle$ state, it is the $\rho_{1,1}$ element of the density matrix. In Fig. 4. we show the numerical results, clearly indicating that the probability tends to the theoretical value of $1/5$ according to (23).

VI. CONCLUSION

In this article we studied the time evolution of the continuous-time quantum walk on a percolation graph. Both unique trajectories and their average properties by using a superoperator formalism have been examined. We found that if the τ time scale of changing the graph tends to zero, then the time evolution can be described with a simple analytically formula. In fact, the percolation causes only a time rescaling compared to the percolation free case. This result is independent of from the underlying graph and we could prove this theorem analytically in the $\tau \to 0$ limes. Naturally, in a real physical system we can expect that τ is a small but finite number. We have presented numerical examples to demonstrate how the system follows our prediction in the long time limit if we choose the τ time steps small enough. The discussed system may serve as a simple model for transport in disordered media.

Acknowledgments

We acknowledge support by the Hungarian Scientific Research Fund (OTKA) under Contract No. K83858, and the Hungarian Academy of Sciences (Lendület Program, LP2011-016).

We thank Prof. Oliver Mülken and Anastasiia Anishchenko for the kind hospitality during a visit in Freiburg and for stimulating discussions.

[1] P. Erdős and A. Rényi, A Matematikai Kutató Intézet Közleményei V. A/1-2. 17-61 (1960).
[2] Béla Bollobás, Random Graphs, 2nd Edition, 2001, Cambridge University Press.
[3] R. Albert and A-L. Barabási, Rev. Mod. Phys. **74**, 47-97, (2002).
[4] M. Newman, A-L. Barabási and D. J. Watts, *The Structure and Dynamics of Networks*, Princeton University Press, (2006).
[5] M. Sahini and M. Sahimi, *Applications Of Percolation Theory*, (CRC Press, 1994).
[6] B. D. Hughes, *Random Walks and Random Environments*, (Oxford University Press, 1995).
[7] Salvador E. Venegas-Andraca, Quantum Information

Processing **11**, 1015-1106 (2012).
[8] N. Konno, in Quantum Potential Theory: Lecture Notes in Mathematics, edited by U. Franz and M. Schurmann (Springer, New York, 2008), Vol. 1954, pp. 9–452.
[9] Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A **48**, 1687 (1993).
[10] E. Farhi and S. Gutmann, Phys. Rev. A **58**, 915 (1998).
[11] A. M. Childs, Phys. Rev. Lett. **102**, 180501 (2009).
[12] O. Mülken, A. Blumen, Physics Reports **502**, 37-87 (2011).
[13] O. Mülken, A. Blumen, Physica E **42**, 576-579, (2010).
[14] G. Leung et al., New J. Phys. **12**, 123018 (2010).
[15] O. Mülken, V. Pernice and A. Blumen, Phys. Rev. E **76**, 051125 (2007).
[16] A. Anishchenko, A. Blumen and O. Mülken, Quantum Information Processing 11, 1273 (2012).
[17] Jeffrey E. Steif, arXiv:0901.4760v2
[18] B. Kollár, T. Kiss, J. Novotný, I. Jex, Phys. Rev. Lett. **108**, 230505 (2012)
[19] D. Meyer, J. Stat. Phys. **85**, 551 (1996), Phys. Lett. A **223**, 337 (1996)
[20] J. Watrous, J. Comput. Syst. Sci. **62**, 371 (2001)
[21] Masuo Suzuki, Commun. Math. Phys. **57**, 193-200 (1977)
[22] J. Geiser, G. Tanoğlu, N. Gücüyenen, Computers and Mathematics with Applications **62** 1994-2015 (2011)
[23] J. Novotný, G. Albert and I. Jex, J. Phys. A: Math. Theor. **42** (2009) 282003
[24] J. Novotný, G. Albert and I. Jex, Cent. Eur. J. Phys. vol. 8, no. 6, pp. 1001-1014, 2010
[25] J. Novotny, G. Alber, I. Jex, New J. Phys. **13**, 053052 (2011)
[26] Z. Darázs and T. Kiss, Phys. Rev. A **81** 062319 (2010)
[27] X. K. Zhang, J. Wan, J. J. Lu, X. P. Xu, Commun. Theor. Phys. **56** (2011) 293-296
[28] J. Wan, X. P. Xu, Physica A 391 (2012) 1919-1927
[29] E. Agliari, A. Blumen and O. Mülken, J. Phys. A: Math. Theor. **41**, 445301 (2008)
[30] O. Mülken, A. Volta, A. Blumen, Phys. Rev. A **72**, 042334 (2005)
[31] O. Mülken and A. Blumen, Phys. Rev. E **71**, 036128 (2005)
[32] B. Kollár, T. Kiss, unpublished.